MODIFICATION OF BLACK-HOLE ENTROPY BY STRINGS

R.Parthasarathy[1] and K.S.Viswanathan[2]
Department of Physics
Simon Fraser University
Burnaby, B.C., Canada V5A 1S6.

ABSTRACT

A generalized action for strings which is a sum of the Nambu-Goto and the extrinsic curvature (the energy integral of the surface) terms, is used to couple strings to gravity. It is shown that the conical singularity has deficit angle that has contributions from both the above terms. It is found that the effect of the extrinsic curvature is to oppose that of the N-G action for the temperature of the black-hole and to modify the entropy-area relation.

[1]Permanent Address: The Institute of Mathematical Sciences, C.I.T.Campus, Taramani Post, Madras 600113,India; e-mail: rparthas@sfu.ca and sarathy@imsc.ernet.in
[2]e-mail: kviswana@sfu.ca

Recently Englert, Houart and Windey [1] obtained a relation between the entropy and the area of a black-hole that is different from the Bekenstein-Hawking [2,3] relation $S = A/4$ ($G = 1$ units) when a conical singularity producing source is present in the euclidean section at $r = 2M$. It is known that a conical singularity at $r = 2M$ modifies the euclidean periodicity of the blackhole metric and hence the temperature [4,5,6]. It is necessary to take into account the source producing the singularity to maintain euclidean saddle point. The authors in Ref.1, introduced for this purpose an elementary string in the action. The conical singularity arises from the string when it wraps around the horizon and the resulting deficit angle is determined by the string tension. In the evaluation of the free energy, the contribution of the string term exactly cancels that of the Einstein term and so only the boundary terms contribute. As a result one obtains,

$$S = \frac{A}{4}(1 - 4\mu) \tag{1}$$

where S is the entropy of the blackhole, A its horizon area and μ the string tension. The string action considered in [1] is the Nambu-Goto action,

$$I_{string} = -\mu \int \sqrt{-g} d\sigma d\tau, \tag{2}$$

where g is the determinant of the induced metric on the worldsheet.

This observation i.e. (1), is important in the light of the recent statistical derivation of Bekenstein-Hawking $S = A/4$ relation in string theory by counting the (microscopic) BPS bound state degeneracy [7,8,9,10,11]. As a result of introducing (2) on the horizon, the temperature of the blackhole is increased as $T = T_H/(1 - 4\mu)$ leading to an acceleration of the evaporation.

The key observation in [1] is the existence of non-trivial solution to the string equations of motion from (2) when the string wraps around the euclidean continuation of the horizon, a sphere at $r = 2M$. When the string worldsheet wraps around the sphere, it has non-zero *extrinsic curvature* and so the simple string action (2) needs a generalization. The Nambu-Goto action is just the area term while the 'energy of the surface' is given by an action involving extrinsic curvature as $\int \sqrt{-g}|H|^2 d\sigma d\tau$. It will be shown that the space-time energy momentum tensor for the Nambu-Goto action

(see the first line in (12)) has the property that there is no flux normal to the surface. On the other hand, the energy momentum tensor for the extrinsic curvature has a non-vanishing flux normal to the surface. Consequently it is conceivable that this flux flow might tend to slow down the acceleration of the evaporation, favouring the stability of the black-hole against the effect in [1]. Indeed our present study confirms this.

The generalized string action [12,13,14] we use in this article is a sum of the N-G action and the one involving extrinsic geometry and is given by,

$$I_{string} \;\; = \;\; -\mu \int \sqrt{-g} d\sigma d\tau + \frac{1}{\alpha_0^2} \int \sqrt{-g} |H|^2 d\sigma d\tau, \tag{3}$$

where $|H|^2 = \sum_i H^i H^i$, $H^i = \frac{1}{2} H^{i\alpha\beta} g_{\alpha\beta}$, $H^{i\alpha\beta}$ is the second fundamental form and i runs from 1 to $D-2$. The string worldsheet is considered as a 2-dimensional surface immersed in a D-dimensional space-time. $H^{i\alpha\beta}$ is then defined by the Gauss equation

$$\partial_\alpha \partial_\beta X^\mu + \tilde{\Gamma}^\mu_{\nu\rho} \partial_\alpha X^\nu \partial_\beta X^\rho - \Gamma^\gamma_{\alpha\beta} \partial_\gamma X^\mu \;\; = \;\; H^i_{\alpha\beta} N^{i\mu}, \tag{4}$$

where $X^\mu(\sigma, \tau)$ are the immersion coordinates of the string worldsheet, $\tilde{\Gamma}$ is the affine connection for the (curved) D-dimensional space-time, Γ is the affine connection on the string worldsheet, calculated using the induced metric

$$g_{\alpha\beta} \;\; = \;\; \partial_\alpha X^\mu \partial_\beta X^\nu h_{\mu\nu}, \tag{5}$$

$h_{\mu\nu}$ being the metric on the D-dimensional space-time, $N^{i\mu}$ are the $(D-2)$ normals to the string worldsheet and α_0^2 is a dimensionless coupling constant.

The present authors have earlier studied [15,16,17,18] the extrinsic geometry of strings in some detail including instanton effects in constant mean curvature ($H \neq 0$) and minimal ($H = 0$) surfaces [18] and recently [19] considered the intrinsic and extrinsic geometrc properties of the string worldsheets in curved space-time background. In this formalism the only string dynamical degrees of freedom are its immersion coordinates $X^\mu(\sigma, \tau)$. It is the purpose of this article to consider the effect of introducing the generalized string action (3) in place of (2) when the string wraps around the euclidean

horizon of a Schwarzschild black hole. We find the entropy-area relation gets modified as

$$S = \frac{A}{4}\frac{(1-4\mu+(\alpha_0 M)^{-2})^2}{(1-4\mu-(\alpha_0 M)^{-2})}. \tag{6}$$

The temperature of the black-hole is found to be
$T = 8\pi\{M(1-4\mu)+(\alpha_0^2 M)^{-1}\}^{-1}$. This feature of M dependence for T is shared by quantum one-loop effects [20].

The Lorentzian action for gravity coupled to matter fields is

$$\begin{aligned} I &= \frac{1}{16\pi}\int\sqrt{-h}Rd^4X - \frac{1}{8\pi}\int_\Sigma\sqrt{-h'}Kd^3X \\ &+ \frac{1}{8\pi}\int_{\Sigma^0}\sqrt{-h^0}K^0 d^3X + I_m, \end{aligned} \tag{7}$$

where the first term is the usual Einstein-Hilbert action, K is the trace of the extrinsic curvature of the boundary Σ of space-time, K^0 is the same as K but of the boundary Σ^0 of flat space-time and I_m is given in (3). The boundary terms in (7) are first introduced by Gibbons and Hawking [21] and their roles in cancelling the boundary terms arising from the Einstein-Hilbert action and removing the divergence at space-like infinity due to the K-term for asymptotically flat space-times are given in [22]. It is to be noted here that we have two extrinsic curvatures; one is the trace of the extrinsic curvature of the boundary of the space-time (denoted by K in (7)) and the other that of the 2-dimensional surface (the string world sheet) immersed in the space-time (denoted by H in (3)) which is not a boundary term. The above action (7) admits the usual Schwarzschild blackhole solution corresponding to trivial solution for I_m i.e., zero string area and vanishing extrinsic curvature. The Euclidean continuation of the Schwarzschild solution is

$$ds^2 = (1-\frac{2M}{r})d\tau^2 + (1-\frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2. \tag{8}$$

Following Hawking [23], we consider an euclidean section via $x = 4M(1-\frac{2M}{r})^{\frac{1}{2}}$, $x \geq 0$, $0 \leq \tau \leq 8\pi M$. The periodicity of τ is $8\pi M$ and this gives,

$$\beta_H = 8\pi M. \tag{9}$$

In the euclidean formalism of blackholes, the topology of the space-time is $R^2 \times S^2$ and with the angular variable τ, the topology becomes $D^2 \times S^2$. There is no cusp now and the deficit angle is zero. The euclidean action following from (7) is

$$\begin{aligned} I_E &= -\frac{1}{16\pi}\int \sqrt{h} R d^4X + \frac{1}{8\pi}\int_{\Sigma} \sqrt{h'} K d^3X - \frac{1}{8\pi}\int_{\Sigma^0} \sqrt{h^0} K^0 d^3X \\ &+ \mu \int \sqrt{g} d^2\sigma - \frac{1}{\alpha_0^2}\int \sqrt{g}|H|^2 d^2\sigma. \end{aligned} \tag{10}$$

Variation of (10) with respect to the background metric $h_{\mu\nu}$ gives the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}h_{\mu\nu}R = -8\pi T_{\mu\nu}, \tag{11}$$

where the energy momentum tensor $T_{\mu\nu}$ for the generalized string action (3) is given by [19]

$$\begin{aligned} T^{\mu\nu} &= \mu \int d^2\sigma \tfrac{\delta^4(X-X(\sigma,\tau))}{\sqrt{h(X(\sigma,\tau))}} \sqrt{g} g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \\ &- \frac{1}{\alpha_0^2}[\int d^2\sigma \sqrt{g} \tfrac{\delta^4(X-X(\sigma,\tau))}{\sqrt{h(X(\sigma,\tau))}}[|H|^2 g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \\ &- g^{\alpha\beta}(\nabla_\beta H^i)(N^{i\nu}\partial_\alpha X^\mu + N^{i\mu}\partial_\alpha X^\nu)] \\ &+ \nabla_\rho \int d^2\sigma \sqrt{g} \tfrac{\delta^4(X-X(\sigma,\tau))}{\sqrt{h(X(\sigma,\tau))}} g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu H^i N^{i\rho}] \end{aligned} \tag{12}$$

In (12) ∇_α is the covariant derivative with respect to the induced metric $g_{\alpha\beta}$ on the worldsheet and ∇_ρ is with respect to the background space-time metric $h_{\mu\nu}$. Note that X in (12) is just the a space-time point, whereas $X(\sigma,\tau)$ stands for the string worldsheet dynamical variables. One sees from the 4-dimensional Dirac delta function in (12) that $T^{\mu\nu}(X)$ vanishes unless X is exactly on the string worldsheet. Further, the flux of $T^{\mu\nu}_{N-G}$, given by the first line in (12) along the normal direction, $T^{\mu\nu}_{N-G}N^i_\nu(X)$ is zero. This is no longer true for that part of $T^{\mu\nu}$ coming from the extrinsic curvature action as can be seen from (12). Variation of (10) with respect to the worldsheet coordinates, $X^\mu(\sigma,\tau) \rightarrow X^\mu(\sigma,\tau) + \delta X^\mu(\sigma,\tau)$ has been evaluated in [19] and by writing

$$\delta X^\mu = \xi_j N^{j\mu} + \xi^\alpha \partial_\alpha X^\mu, \tag{13}$$

as normal and tangential variations, it has been shown that the equation of motion for normal variation is

$$\begin{aligned}\nabla^2 H^i - 2H^i(H^2+k) + H^j H^{j\alpha\beta} H^i_{\alpha\beta} \\ -g^{\alpha\beta} H^j \tilde{R}_{\mu\nu\rho\sigma}\partial_\alpha X^\nu \partial_\beta X^\rho N^{j\mu} N^{i\sigma} &= 0,\end{aligned} \tag{14}$$

where $k = \alpha_0^2/\mu$, $i = 1,2$, $\tilde{R}_{\mu\nu\rho\sigma}$ is the Ricci tensor of the curved background space. The tangential variation has been shown to be just the contracted structure equation of Codazzi. It is clear from (14) that the string worldsheet is non-trivial on the euclidean section alluded to above, as it can admit solutions $H^i \neq 0$. Taking the trace of (11) and integrating over d^4X we find,

$$\int d^4X\sqrt{h}R(X) = 16\pi\{\mu A - \frac{1}{\alpha_0^2}\int d^2\sigma\sqrt{g}|H|^2\}, \tag{15}$$

where A is the area of the string worldsheet. The above result is general in the sense that we have not restricted the string worlsheet to have constant mean curvature.

When the string wraps around the euclidean horizon, a sphere at $r = 2M$, $R \neq 0$ and (14) can have non-trivial solution. In such a case, the integral on the left hand side of (15) gets contribution from the singularity at $r = 2M$. Following [1] and [6], we consider an infinitesimal tubular neighbourhood $D^2 \times S^2$ of $r = 2M$.Then

$$\int d^4X\sqrt{h}\ R(X) = A\int_{D^2}\sqrt{{}^{(2)}h}\ {}^{(2)}R\ d^2X, \tag{16}$$

a product of A, the area of S^2, (identified as the horizon area) and the Euler characteristic of disk. From (15) and (16) it follows that

$$\frac{1}{4\pi}\int_{D^2}\sqrt{{}^{(2)}h}\ {}^{(2)}R\ d^2X = 4\{\mu - \frac{1}{\alpha_0^2 A}\int d^2\sigma\sqrt{g}|H|^2\}. \tag{17}$$

But, by Gauss-Bonnet theorem, the left hand side of (17) is the Euler characteristic χ of the disk which is 1. Thus the introduction of generalized string on the horizon has changed the topology of the disk by creating a cusp. The deficit angle is

$$\eta = 4\{\mu - \frac{1}{\alpha_0^2 A}\int d^2\sigma\sqrt{g}|H|^2\} \tag{18}$$

Comparison with Eqn.16 of [1] shows that the generalized string with extrinsic curvature has a non-trivial effect. For a sphere the mean curvature is a constant [24], $|H|^2 = 1/r^2$. The effect of adding the extrinsic curvature action is thus to modify the periodicity of τ from $\beta_H = 8\pi M$ to

$$\beta = \beta_H\{1 - 4\mu + (\alpha_0 M)^{-2}\}, \tag{19}$$

where $r = 2M$ is used. It follows from (19) that in the presence of the extrinsic curvature action, the increase in the temperature of the black-hole due to the N-G action is reduced so that the acceleration of evaporation is slowed down.

The calculation of the free energy of the black-hole is made a lot easier in view of (15). Accordingly the contribution from the Einstein term $\int \sqrt{h} R d^4 X$ in (10) is exactly cancelled by the string contribution including the action for the extrinsic curvature. Thus the only contribution to the free energy comes from the boundary terms in (10). This contibution has been evaluated by Gibbons and Hawking [21] (see also [1]) as

$$I_{Boundary} = \frac{\beta^2}{16\pi}, \tag{20}$$

from which the free energy ($\beta^{-1}\ I_{Boundary}$) is

$$E_{free} = \frac{M}{2}. \tag{21}$$

The free energy is the same without the generalized string action. From (19), using (9) we find,

$$\frac{dM}{d\beta} = \{8\pi(1 - 4\mu - (\alpha_0^2 M^2)^{-1}\}^{-1}, \tag{22}$$

and using the relation for entropy $S = \beta^2 \frac{dE_{free}}{d\beta}$ we have

$$S = \frac{A}{4}\frac{\{1 - 4\mu + (\alpha_0 M)^{-2}\}^2}{\{1 - 4\mu - (\alpha_0 M)^{-2}\}}. \tag{23}$$

Thus the in the presence of string wrapping the event horizon, the entropy differs from that of the Bekenstein-Hawking $A/4$ relation.

Let us summarize our results. We have argued that when the string worldsheet wraps around the horizon, a 2-sphere ($r = 2M$), it has extrinsic curvature and this has been added to the Nambu-Goto term in the form of 'energy integral' of the surface. The Einstein equations of motion with the modified energy momentum tensor give the relation (15) from which it follows that the free energy is independent of the string contribution as it is cancelled by the Einstein term. By considering infinitesimal tubular neighbourhood of $r = 2M$, the conical singularity has a deficit angle (18). It is to be noted that the deficit angle depends on the black-hole mass in contrast to the situation in [1]. This changes the periodicity of τ and change the entropy-area relation according to (23). When the mean curvature is constant, it has been shown [18] that the string admits instanton solutions. Its effect in the context of QCD was studied in [18]. In this article another effect of string instanton is described. In the absence of extrinsic curvature, the string instanton has the effect of raising the global temperature of the black-hole. The extrinsic curvature action has the opposite effect as far as the temperature is concerned (19). It is intresting to note that the inverse temperature β of the black-hole has M dependence through the N-G action and M^{-1} dependence through the extrinsic curvature action. In [20] a similar feature was obtained from the one-loop quantum effects, M from the classical action and M^{-1} from the quantum correction. The extrinsic curvature action can arise as quantum correction when fermions on the string world sheet are functionally integrated [25]. It can be seen that the sign of σ in [20] for spin-half fields is negative agreeing with our $\alpha_0^2 > 0$. The entropy-area relation is modified. The black-hole entropy can be greater that $A/4$. It is possible to interpret (23) within the $S = A/4$ relation if we introduce an effective area for the horizon as

$$A_{eff} = A\{1 - 4\mu + (\alpha_0 M)^{-2}\}^2 \{1 - 4\mu - (\alpha_0 M)^{-2}\}^{-1}. \qquad (24)$$

It can be seen that $A_{eff} > A$ and so that A_{eff} represents a 'stretched horizon'. In the context of extremal black-holes, the degeneracy associated with extremal black-hole states is smaller than the degeneracy of the elementary string states with the same quantum numbers. To resolve this, Sen [26] postulated that the entropy of the extremal black-hole is not exactly equal to the area of the event horizon, but the area of a surface close to the event horizon; the 'stretched horizon'. It is interesting to see that the introduction

of extrinsic curvature action for strings on the horizon favours this idea of 'stretched' horizon for neutral Schwarzschild black-hole as well.

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